2024 Korean Student Combinatorics Workshop (KSCW2024)

공주 한옥마을

July 29 - August 2, 2024

Information

- Title: 2024 Korean Student Combinatorics Workshop, 2024 조합론 학생 워크샵
- Date: July 29 - August 2, 2024
- Venue: 공주 한옥마을
- Website: https://indico.ibs.re.kr/event/651/

Invited Speakers

- Semin Yoo (IBS DIMAG)
- Jungho Ahn (KIAS)

Contributed Speakers

- Homoon Ryu (Seoul National University)
- Jaehyeon Seo (Yonsei University)
- Dohyeon Lee (KAIST & IBS DIMAG)

Organizers

- Donggyu Kim (KAIST & IBS DIMAG)
- Seokbeom Kim (KAIST & IBS DIMAG)
- Seonghyuk Im (KAIST & IBS ECOPRO)
- Hyunwoo Lee (KAIST & IBS ECOPRO)

TIMETABLE

July 29 (Monday)

Time	Speaker	Schedule
13:00-14:00		등록
14:00-14:20		개회 및 인사말
14:20-16:20		자기소개
16:30-17:30	Semin Yoo	**Invited Talk:** A postdoc life in Korea
17:30-18:20		Open problem session
18:30-		Dinner (율화관, 소불고기정식)

July 30 (Tuesday)

Time	Speaker	Schedule
09:30-10:30	Dohyeon Lee	**Contributed Talk:** Colorful intersections and Tverberg partitions
11:00-12:00	Jaehyeon Seo	**Contributed Talk:** Transversal Hamilton paths and cycles of arbitrary orientations in tournaments
12:00-14:00		Lunch (율화관, 연잎밥정식)
14:00-15:00		Open problem session
15:30-18:00		Working session
19:30-		Dinner (한옥마을 BBQ)

July 31 (Wednesday)

Time	Speaker	Schedule
09:30-10:30	Homoon Ryu	**Contributed Talk:** How to determine a graph is Toeplitz?
11:00-12:00		Working session
12:00-14:00		Lunch (한옥관, 곰탕 및 한치물회)
14:00-15:00		Progress report
15:00-16:00	Jungho Ahn	**Invited Talk:** 학술적 자기 PR 방법
16:15-18:00		Working session
18:00-		Dinner (새이학가든, 공주국밥정식)

August 1 (Thursday)

Time	Speaker	Schedule
09:00-12:00		Excursion: 공산성 산책
12:00-14:00		Lunch
14:00-15:00		Progress report
15:15-18:00		Working session
18:00-		Dinner (예가촌, 돼지석갈비)

August 2 (Friday)

Time	Speaker	Schedule
09:00-10:00		Working session
10:00-11:30		Progress report
11:30-		맺음말 및 폐회

INVITED TALK

A postdoc life in Korea

July 29
16:30-17:30

Semin Yoo
IBS DIMAG

In this talk, I will talk about my experience of being a postdoc in Korea. In addition, I will discuss how the job process in Korea is going and share some useful advice to increase the possibility of getting a tenure-track job even though I have not made it yet.

July 31
15:00-16:00

학술적 자기 PR 방법

Jungho Ahn

KIAS

이 발표에서는 학술적 성취를 효과적으로 표현하고 홍보하기 위한 방법들을 소개합니다. 학술적 웹사이트 생성 방법, CV와 Research statement 작성 및 유의점, 학회에서의 네트워킹 및 학술적 발표 구성 방법 등을 다룹니다.

CONTRIBUTED TALK

Colorful Intersections and Tverberg Partitions

July 30
09:30-10:30

Dohyeon Lee
KAIST & IBS DIMAG

In this talk, I will introduce fundamental theorems in convexity problems of discrete geometry, including Radon's theorem, Tverberg's theorem, Carathéodory's theorem along Helly's theorem along with their colorful versions. I will cover some proofs and techniques such as linear dependency, Sarkaria's tensor product method, configuration space/test map scheme and discrete Morse theory. I will also discuss some transversal theorems, including our own result and the ideas of proof, if time permits.

July 30
11:00-12:00

Transversal Hamilton paths and cycles of arbitrary orientations in tournaments

Jaehyeon Seo

Yonsei University

It is well-known that a tournament always contains a directed Hamilton path. Rosenfeld conjectured that if a tournament is sufficiently large, it contains a Hamilton path of any given orientation. This conjecture was approved by Thomason, and Havet and Thomassé completely resolved it by showing there are exactly three exceptions.

We generalized this result into a transversal setting. Let $\mathbf{T} = \{T_1, \ldots, T_{n-1}\}$ be a collection of tournaments on a common vertex set V of size n. We showed that if n is sufficiently large, there is a Hamilton path on V of any given orientation which is obtained by collecting exactly one arc from each T_i. Such a path is said to be *transversal*.

It is also a folklore that a strongly connected tournament always contains a directed Hamilton cycle. Rosenfeld made a conjecture for arbitrarily oriented Hamilton cycles in tournaments as well, which was approved by Thomason (for sufficiently large tournaments) and Zein (by specifying all the exceptions). We also showed a transversal version of this result. Together with the aforementioned result, it extends our previous research, which is on transversal generalizations of existence of directed paths and cycles in tournaments.

This is a joint work with Debsoumya Chakraborti, Jaehoon Kim, and Hyunwoo Lee.

How to determine a graph is Toeplitz?

July 31
09:30-10:30

Homoon Ryu

Seoul National University

A given graph is called Toeplitz if it has the Toeplitz matrix as its adjacency matrix. I proposed a problem that how can we determine a given graph is a Toeplitz graph or not. For this purpose, I give a talk about Toeplitz graphs that might be related to this problem. Some simple observations and some previous results about Toeplitz graphs will be covered.

OPEN PROBLEMS

Finding two disjoint anticomplete cycles

Jungho Ahn
KIAS

Prove or disprove that in polynomial time, we can find two cycles such that they have disjoint vertex sets and there is no edge between them.

Characterize the Degree Constraint for the Subgraph Complement Problem

Shinwoo An
POSTECH

Let $X = (A \cup B, A \times B)$ be a bipartite graph under two sets A, B of size n. Suppose there is a bipartite graph $G \subset X$, which is not given explicitly, but we know the degree $\deg_G(v)$ for all $v \in A \cup B$. Moreover, suppose there is another bipartite graph $H \subset X$, which is explicitly given, and we know that $\deg_H(v) \leqslant \deg_G(v)$ for every $v \in A \cup B$.

Here is the problem.

Question 1. *Does there exists a bipartite graph $I \subset X$ such that $deg_I(v) = deg_G(v) - deg_H(v)$ for all v?*

Of course, this does not holds in general: If $\deg_G(v) = 2$ for all $A \cup B$ and $\deg_H(v) = 2$ for all but two vertices of $A \cup B$ and $\deg_H(v) = 0$ for those two vertices, we cannot construct I. Therefore, here is the open problem.

Problem 2. *Classify the degree constraint for G and H so that there exists a bipartite graph I such that $deg_I(v) = deg_G(v) - deg_H(v)$ for all $v \in A \cup B$.*

We can think about the stronger version.

Problem 3. *Classify the degree constraint for G and H so that there exists a bipartite graph I such that $deg_I(v) = deg_G(v) - deg_H(v)$ for all $v \in A \cup B$. Furthermore, we impose the condition that H is given explicitly and H, I are edge-disjoint graphs.*

Characterizing the class of graphs of radius-1 flip-width at most 2

Yeonsu Chang
Hanyang University

Toruńczyk introduced a graph parameter called radius-r flip-width for $r \in \mathbb{N} \cup \{\infty\}$, which is defined using a variant of the cops and robber game, so called a flipper game.

Let G be a graph and let $\mathcal{P}$ be a partition of the vertex set of G. We say that G' is a *$\mathcal{P}$-flip* of G if $G' = G \oplus \mathcal{S}$ with $\mathcal{S} = \{(A_i, B_i) : i \in I\}$ such that for all $i \in I$, $A_i, B_i \in \mathcal{P}$. Since flips are involutive and commute with each other, we may assume that $\mathcal{S}$ contains at most $\binom{|\mathcal{P}|+1}{2}$ pairs. We say that a $\mathcal{P}$-flip G' of G is a *k-flip* if $|\mathcal{P}| \leqslant k$.

Let $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$. The *flipper game* with radius r and width k is played on a graph G. At the beginning, set $G_0 = G$ and the robber selects a starting vertex v_0 of G. In each i-th round for $i > 0$, cops announce a k-flip G_i of G and the robber knows G_i and selects a new position $v_i \in V(G)$ following a path of length at most r from v_{i-1} in the previous graph G_{i-1}. The game terminates when the robber is caught, meaning that v_i is isolated in G_i.

The *radius-r flip-width* of a graph G, denoted by $\mathrm{fw}_r(G)$, is the minimum k such that the cops have a winning strategy in the flipper game of radius r and width k on G.

We showed that C_5, bull, gem and co-gem have radius-r flip-width at least 3 for each $r \in (\mathbb{N} \setminus \{1\}) \cup \{\infty\}$ and (C_5, bull, gem, co-gem)-free graphs have radius-r flip-width at most 2 for each $r \in \mathbb{N} \cup \{\infty\}$.

Theorem 4. *Let G be a (C_5, bull, gem, co-gem)-free graph and $r \in \mathbb{N} \cup \{\infty\}$. Then $fw_r(G) \leqslant 2$.*

We observe that gem and co-gem have radius-1 flip-width at most 2. Thus, the class of graphs of radius-1 flip-width at most 2 is strictly larger than the class of (C_5, bull, gem, co-gem)-free graphs.

Question 5. *Characterize the class of graphs of radius-1 flip-width at most 2 in terms of forbidden induced subgraphs.*

References

[1] Szymon Toruńczyk. Flip-width: Cops and robber on dense graphs. arXiv:2302.00352, 2023
[2] Yeonsu Chang, Sejin Ko, O-joung Kwon, and Myounghwan Lee. A characterization of graphs of radius-r flip-width at most 2. 2023. arXiv:2306.15206.

The Lower bounds for (Multicut) Mimicking Network

Kyungjin Cho
POSTECH

Mimicking problem for (G, T) aims to find a minor graph of G maintaining the minimum edge cut size between two parts $(A, T - A)$ for any $A \subset T$. Additionally, in the context of separating terminals into more than two parts, there exists a corresponding graph sparsification problem known as the *multicut-mimicking problem*. A *multicut-mimicking network* for terminals T in a normal graph G is a minor graph that preserves the size of the minimum multicut of any set of cut requests over T. Equivalently, it preserves the size of the minimum multiway cut of any partition of T.

It is already known that there exists a planar graph G and terminal set $T \subset V(G)$ where every mimicking network of (G, T) has at least $2^{|T|-2}$ edges [1]. Especially, there is a mimicking network of (G, T) with $\Theta(|T|^2)$ edges for any planar graph G and terminals $T \subset V(G)$ if terminals in T are located on the same face, which is optimal [2, 3]. However, parameterized by $k = \sum_{t \in T} \mathsf{deg}_G(t)$, it was represented that an instance (G, T) has a mimicking network with at most $O(k^4)$ edges for a general hypergraph G [4, 5]. Furthermore, there is a multicut-mimicking network with $k^{O(\log k)}$ hyperedges if G is a normal graph (not hypergraph) [6].

Problem 6. *Is there an instance (G, T) so that every multicut-mimicking network has at least $k^{\Omega(r)}$ hyperedges, where r is the rank of G which is the maximum cardinality of an hyperedge in G?*

References

[1] Karpov, Nikolai, Marcin Pilipczuk, and Anna Zych-Pawlewicz. "An exponential lower bound for cut sparsifiers in planar graphs." *Algorithmica 81* (2019): 4029-4042.

[2] Goranci, Gramoz, Monika Henzinger, and Pan Peng. "Improved guarantees for vertex sparsification in planar graphs." *SIAM Journal on Discrete Mathematics* 34.1 (2020): 130-162.

[3] He, Zhiyang, Jason Li, and Magnus Wahlström. "Near-Linear-Time, Optimal Vertex Cut Sparsifiers in Directed Acyclic Graphs." *29th Annual European Symposium on Algorithms* (ESA 2021).

[4] Liu, Yang P. "Vertex Sparsification for Edge Connectivity in Polynomial Time." 14th *Innovations in Theoretical Computer Science Conference* (ITCS 2023). Vol. 251.

[5] Jiang, Han, et al. "Vertex Sparsifiers for Hyperedge Connectivity." 30th *Annual European Symposium on Algorithms* (ESA 2022).

[6] Wahlström, Magnus. "Quasipolynomial multicut-mimicking networks and kernels for multiway cut problems." *ACM Transactions on Algorithms* (TALG)18.2 (2022): 1-19.

Minimally asymmetric matroids

Mujin Choi

KAIST & IBS DIMAG

In 1988, Nešetřil conjectured that there are only finite number of minimally asymmetric graphs. In 2016, Pascal Schweitzer and Patrick Schweitzer confirmed that there are exactly 18 minimally asymmetric graphs. We will consider the matroid version of this problem.

Let M be a matroid. We say that M is *asymmetric* if $Aut(M) = 1$. We say that M is *minimally asymmetric* if no minor of M is asymmetric.

Problem 7. *Is there an infinite number of asymmetric matroids? What if we restrict to representable matroids? If there are only finite number of them, what are they?*

Chracterizing the digraph in which every t vertices have exactly λ common out-neighbors

Hojin Chu

Seoul National University

In 1966, Erdős proved the "Friendship Theorem" that states a (finite) graph in which each pair of vertices has exactly one common neighbor has a vertex adjacent to all the other vertices. Such a graph is called a *friendship graph*. Follow-up studies have been actively conducted by many researchers and the results are as follows: a graph in which every t vertices have exactly λ common neighbors is regular if $t = 2$ and $\lambda \geqslant 2$; the complete graph on $t + \lambda$ vertices if $t \geqslant 3$.

Some researchers tried to characterize the digraphs in which every t vertices have exactly λ common out-neighbors. With my co-researchers, I showed that a digraph in which each pair of vertices has exactly one common out-neighbor, called a liking digraph, is diregular or a fanch wheel digraph, where a *fancy wheel digraph* is obtained from the disjoint union of directed cycles by adding one vertex with arcs to and from each vertex on the cycles. This result extends the Friendship Theorem since, in a symmetric case, it becomes a friendship graph if each directed cycle of length two is replaced with an edge. We call such a digraph a *liking digraph*. In general, a (t, λ)*-liking digraph* is a digraph in which every t vertices have exactly λ common out-neighbors. We showed that a (t, λ)-liking digraph is the complete graph on $t + \lambda$ vertices if $t \geqslant \lambda + 1$ and $t \neq 2$.

Problem 8. *Characterize* (t, λ)*-liking digraphs for* $3 \leqslant t < \lambda + 1$*, or* $t = 2$ *and* $\lambda \geqslant 2$*.*

We also tried to characterize the digraphs in which every t vertices share exactly λ out-neighbors and λ in-neighbors. We call such a digraph a *two-way* (t, λ)*-liking digraph*. We showed that a two-way (t, λ)-liking digraph is diregular if $t = 2$ and $\lambda \geqslant 2$; the complete digraph on $t + \lambda$ vertices if $t \geqslant 3$. The results extend the follow-up studies of the Friendship Theorem. Additionally, these implies that for $t \geqslant \lambda + 1$, in a digraph, every t vertices have exactly λ common in-neighbors if and only if every t vertices have exactly λ common out-neighbors.

Problem 9. *Indentify* t *and* λ *so that, in a digraph, every* t *vertices have exactly* λ *common in-neighbors if and only if every* t *vertices have exactly* λ *common out-neighbors.*

Grids and cylinders in a graph with many copies of C_4

Seonghyuk Im

KAIST & IBS DIMAG

Generalized Turán number $\mathrm{ex}(n, F, H)$ is the maximum number of F copies in an n-vertex graph G which does not contain H as a subgraph. When $F = K_2$, it is the same as the classical Turán number, denoted by $\mathrm{ex}(n, H)$.

In 2022, Bradač, Janzer, Sudakov, and Tomon [1] proved that for any fixed t, $\mathrm{ex}(n, t \times t \text{ grid}) = O(n^{3/2})$. In 2023, Gao, Janzer, Liu, and Xu [3] gave a simpler proof of this result. They also proved the following.

Theorem 10. *Let $C_{2\ell}^{\square}$ be a graph obtained by starting with two disjoint copies of $C_{2\ell}$ and add a perfect matching between the corresponding vertices. Then for every $\ell \geqslant 4$,*

$$\mathrm{ex}(n, C_{2\ell}^{\square}) = O(n^{3/2}).$$

Both of grid and $C_{2\ell}^{\square}$ are obtained by attaching C_4s iteratively. Thus, we want to ask whether the number of C_4 is an essential parameter that guarantees such structures.

Question 11. $\mathrm{ex}(n, C_4, H) = O(n^2)$ *when H is $t \times t$ grid or $C_{2\ell}^{\square}$.*

We note that the solution of this question gives an alternative proof of classical Turán problem as Erdős and Simonovits [2] proves that if G has $Cn^{3/2}$ edges for some $C > 10$, then G has at least $C^4 n^2/2$ copies of C_4.

References

[1] Domagoj Bradač, Oliver Janzer, Benny Sudakov, István Tomon. (2023). The Turán number of the grid. *Bulletin of the London Mathematical Society*, 55(1), 194-204.

[2] Paul Erdős, Miklós Simonovits. (1983). Supersaturated graphs and hypergraphs. *Combinatorica, 3*, 181-192.

[3] Jun Gao, Oliver Janzer, Hong Liu, Zixiang Xu. (2023). Extremal number of graphs from geometric shapes. *arXiv preprint arXiv:2303.13380.*

Comparing numbers of matchings in two bipartite graphs

Donggyu Kim
KAIST & IBS DIMAG

Hall's marriage theorem states that for a bipartite graph G with a vertex bipartition (A, B), if $|N_G(X)| \geqslant |X|$ for all $X \subseteq A$, then G has a matching covering A. So, it is natural to ask the following question, which I heard from a workshop held in Okinawa [1].

Problem 12. *Let G_1 and G_2 be bipartite graphs with bipartitions (A, B_1) and (A, B_2). If $|N_{G_1}(X)| \geqslant |N_{G_2}(X)| \geqslant |X|$ for every $X \subseteq A$, then is the number of matchings covering A in G_1 is larger than or equal to that in G_2?*

References

[1] The 3rd East Asia Workshop on Extremal and Structural Graph Theory (November 1–5, 2023), Okinawa, Japan, https://tgt.ynu.ac.jp/2023EastAsia.html

$X + Y$ sorting

Juwon Kim
KAIST

Given two sets of numbers, each of size n, how quickly can the set of all pairwise sums be sorted? In symbols, given two sets X and Y, our goal is to sort the set

$$X + Y = \{x + y \mid x \in X, y \in Y\}.$$

Problem 13. *Is there an $X + Y$ sorting algorithm faster than $\mathcal{O}(n^2 \log n)$?*

The obvious $O(n^2 \log n)$-time algorithm is also the fastest known. There are $\Omega(n^2)$ lower bounds for this problem in various restrictions of the linear decision tree model of computation [7, 4, 6]. The main problem is whether the logarithmic factor can be removed.

Fredman [7] proved that if a given partial order on m elements has L linear extensions, then the set can be sorted in at most $\log_2 L + 2m$ comparisons. For the sorting $X + Y$ problem, we have $m = n^2$, the Hasse diagram of the partial order is an $n \times n$ diagonal grid, and simple arguments about hyperplane arrangements imply that $L = O(n^{8n})$. Thus, Fredman's algorithm can sort $X + Y$ using only $8n \log n + 2n^2$ comparisons; unfortunately, the algorithm needs exponential time to choose which comparisons to perform! This exponential overhead was reduced to polynomial time by Kahn and Kim [8] and then to $O(n^2 \log n)$ by Lambert [9] and Steiger and Streinu [10]. These results imply that no superquadratic lower bound is possible in the full linear decision tree model.

If the input consists of n integers between $-M$ and M, an algorithm of Seidel based on fast Fourier transforms runs in $O(n + M \log M)$ time [6]. The $\Omega(n^2)$ lower bounds require exponentially large integers.

A closely related problem does have a subquadratic solution: find a minimum element of $X + Y$, the so-called min-convolution problem, posed by Jeff Erickson [5]. See [2] for the result and a discussion of connections to the sorting problem.

References

[1] A. Hernández Barrera. Finding an $O(n^2 \log n)$ algorithm is sometimes hard. In *Proc. 8th Canad. Conf. Comput. Geom.*, pages 289–294. Carleton University Press, Ottawa, Canada, 1996.

[2] David Bremner, Timothy M. Chan, Erik D. Demaine, Jeff Erickson, Ferran Hurtado, John Iacono, Stefan Langerman, and Perouz Taslakian. In *Proceedings of the 14th Annual European Symposium on Algorithms (ESA 2006)*, pages 160–171, Zürich, Switzerland, September 11–13 2006.

[3] G. Barequet and S. Har-Peled. Polygon containment and translational min-Hausdorff-distance between segment sets are 3SUM-hard. *Internat. J. Comput. Geom. Appl.*, 11:465–474, 2001.

[4] M. Dietzfelbinger. Lower bounds for sorting of sums. *Theoret. Comput. Sci.*, 66:137–155, 1989.

[5] Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2005. In *Proc. 18th Canad. Conf. Comput. Geom.*, pages 75–80, 2006.

[6] Jeff Erickson. Lower bounds for linear satisfiability problems. *Chicago J. Theoret. Comput. Sci.*, 1999(8), 1999.

[7] M. L. Fredman. How good is the information theory bound in sorting? *Theoret. Comput. Sci.*, 1:355–361, 1976.

[8] Jeff Kahn and Jeong Han Kim. Entropy and sorting. *J. Comput. Sys. Sci.*, 51:390–399, 1995.

[9] Jean-Luc Lambert. Sorting the sums $x_i + y_i$ in $O(n^2)$ comparisons. *Theoret. Comput. Sci, 103:137-141, 1992.*

[10] W. Steiger and Ileana Streinu. A pseudo-algorithmic separation of lines from pseudo-lines. *Inform. Process. Lett.*, 53:295–299, 1995.

Polynomial χ-boundedness, Pollyanna class, and self-isolation

Seokbeom Kim
KAIST & IBS DIMAG

For graphs G and H, a *copy* of H in G is an induced subgraph of G isomorphic to H, and we say G is *H-free* if it does not have a copy of H. A class of graphs is *hereditary* if it is closed under taking induced subgraphs. A hereditary class of graphs $\mathcal{C}$ is *χ-bounded* if there is a function f such that $\chi(G) \leqslant f(\omega(G))$ for every $G \in \mathcal{C}$, where $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G, respectively. Such a function f is called a *χ-bounding function* for $\mathcal{C}$. In particular, if $\mathcal{C}$ admits a polynomial χ-bounding function, we say $\mathcal{C}$ is *polynomially χ-bounded.*

In the theory of χ-boundedness, there is a long-standing conjecture by Gyárfás and Sumner that is still widely open.

Conjecture 14 (Gyárfás [5], Sumner [8])**.** *For a graph H, the class of H-free graphs is χ-bounded if and only if H is a forest.*

The only if part of the conjecture directly follows from the following theorem by Erdös.

Theorem 15 (Erdös [4])**.** *For integers $g, k \geqslant 3$, there are graphs of chromatic number at least k that do not contain cycles of length less than g.*

It suffices to prove the Gyárfás-Sumner conjecture when H is a tree, as a forest satisfies the conjecture if and only if each component does. However, things get much harder when we consider polynomial χ-boundedness. Say a forest F is *good* if the class of F-free graphs is polynomially χ-bounded.

Question 16. *If two forests F_1 and F_2 are good, is the graph $F_1 \cup F_2$ also good?*

(Here, $F_1 \cup F_2$ denotes the disjoint union of F_1 and F_2.)

On the other hand, Chudnovsky, Cook, Davies, and Oum [1] recently introduced the notion of Pollyanna classes of graphs, which is a nice generalization of polynomial χ-boundedness. A hereditary class $\mathcal{C}$ of graphs is *Pollyanna* if $\mathcal{C} \cap \mathcal{F}$ is polynomially χ-bounded for every χ-bounded class $\mathcal{F}$ of graphs. We can again propose a similar question for Pollyanna classes of graphs: Say a graph H is *nice* if the class of H-free graphs is Pollyanna.

Question 17. *If two graphs H_1 and H_2 are nice, is the graph $H_1 \cup H_2$ also nice?*

Observe that every polynomially χ-bounded class is Pollyanna, but the converse is not true (consider the class of K_t-free graphs). Thus, Question 17 asks for a more general property than Question 16. Somewhat surprisingly, there is a notion that can give positive answers to both questions.

A graph J is *self-isolating* if for every non-decreasing polynomial φ, there is a polynomial ψ with the following property. For every graph G with $\chi(G) > \psi(\omega(G))$, there exists $A \subseteq V(G)$ with $\chi(A) > \varphi(\omega(A))$ such that either

- $G[A]$ is J-free, or
- G contains an copy J' of J such that $V(J')$ is disjoint from and anticomplete to A.

(Here, $\chi(A) = \chi(G[A])$, $\omega(A) = \omega(G[A])$, and two sets $X, Y \subseteq V(G)$ are *anticomplete* if there are no edges with one end in X and the other end in Y.)

The self-isolation of a graph is a notion that fits well with taking the disjoint union of good or nice graphs. Indeed, if both graphs H and J are good (respectively, nice) and J is self-isolating, then $H \cup J$ is also good (respectively, nice).

Now, we return to the original question: what forests are good? Clearly, stars are good (by Ramsey's theorem), and P_4 is also good (as P_4-free graphs are cographs, which are perfect). However, until recently, it was unknown whether the disjoint union of stars and P_4s is good. In a series of papers, Scott, Seymour, and Spirkl proved that (see [6, 2, 7])

- stars are self-isolating;
- (with Chudnovsky) the path P_4 on 4 vertices is self-isolating; and
- every P_5-free tree is good.

Under this direction, there is a question that remains to be asked.

Problem 18. *Is every P_5-free tree self-isolating?*

The smallest open case is the following. A *fork* is a graph obtained from $K_{1,3}$ by subdividing an edge exactly once.

Problem 19. *Is a fork self-isolating?*

Without considering the context of polynomial χ-boundedness, one can also ask which graphs (not necessarily forests) are self-isolating. Chudnovsky, Scott, Seymour, and Spirkl [3] showed that cliques and complete bipartite graphs are self-isolating.

Problem 20. *Is the notion of self-isolation closed under taking induced subgraphs? In detail, suppose that a graph H is self-isolating and H' is an induced subgraph of H. Is it true that H' is also self-isolating?*

Problem 21. *If Problem 20 is true, which graphs are minimally non-self-isolating? Can we fully characterize self-isolating graphs by their forbidden induced subgraphs?*

References

[1] Maria Chudnovsky, Linda Cook, James Davies, and Sang-il Oum (2023). Reuniting χ-boundedness with polynomial χ-boundedness. arXiv:2310.11167.

[2] Maria Chudnovsky, Alex Scott, Paul Seymour, and Sophie Spirkl (2023a). Polynomial bounds for chromatic number. VI. Adding a four-vertex path. *European Journal of Combinatorics.* **110**, No. 103710, 10.

[3] Maria Chudnovsky, Alex Scott, Paul Seymour, and Sophie Spirkl (2023b). Polynomial bounds for chromatic number. VII. Disjoint Holes. *J. Graph Theory.* **104**, 499-515.

[4] Paul Erdös (1959). Graph theory and probability. *Canadian Journal of Mathematics.* **11**, 34-38.

[5] András Gyárfás (1975). On Ramsey covering-numbers. *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdös on his* 60*th birthday), Vol. II,* 801-816. Colloq. Math. Soc. Janos Bolyai **10**.

[6] Alex Scott, Paul Seymour, and Sophie Spirkl (2022a). Polynomial bounds for chromatic number. II. Excluding a star forest. *J. Graph Theory.* **101**, 318-322.

[7] Alex Scott, Paul Seymour, and Sophie Spirkl (2022b). Polynomial bounds for chromatic number. III. Excluding a double star. *J. Graph Theory.* **101**, 323-340.

[8] David P. Sumner (1981). Subtrees of a graph and the chromatic number. *The theory and applications of graphs (Kalamazoo, Mich., 1980),* 557-576.

Geometric join of more than d sets in $\mathbb{R}^d$ is contratible

Dohyeon Lee
KAIST & IBS DIMAG

Convex hull of a set A is the smallest convex set containing X and denoted by conv$X$. For nonempty subsets $X_1, \ldots, X_m$ of $\mathbb{R}^d$, their geometric join $X_{[m]}$ is defined by

$$X_{[m]} = \bigcup_{x_i \in X_i} \mathrm{conv}\{x_1, \ldots, x_m\} = \{\sum t_i x_i : \sum t_i = 1, t_i \in [0,1], x_i \in X_i\}.$$

Each class X_is are called *color classes*, a subset $Y \subseteq \cup X_i$ is called *colorful* if $|Y \cap X_i| \leqslant 1$ for each i and its convex hull is referred *colorful simplex*. In this language, geometric join of $X_1, \ldots, X_m$ is union of *colorful simplex*.

Conjecture 22. *The geometric join $X_{[m]}$ is contractible if $m \geqslant d+1$.*

Theorem 23. [1] *Conjecture 22 is true for $d = 2$ and $d = 3$.*

If some color class is singleton, then geometric join is star-shaped, so contractible. The problem is open even for the case $m = d+1$ and $|X_i| = 2$ for each i. But there is some result when m is large.

Theorem 24. [1] *Conjecture 22 is true for $m > \frac{d(d+1)}{2}$. Moreover for those m, $X_{[m]}$ is star-shaped.*

References

[1] Bárány, I., Holmsen, A. & Karasev, R. Topology of Geometric Joins. *Discrete & amp; Computational Geometry.* **53**, 402-413 (2015,2), http://dx.doi.org/10.1007/s00454-015-9665-2

Random matchings

Hyunwoo Lee

KAIST & IBS ECOPRO

We say a k-uniform hypergraph (k-graph) H is a linear hypergraph if every distinct pair of edges of H shares at most one vertex in common. For a vertex $v \in V(H)$, denote $\mathcal{P}_H(\overline{v})$ by the probability that a random matching M does not cover v, where M is chosen uniformly at random from the set of matchings of H.

In 1995, Kahn [1, 2] conjectured the following.

Conjecture 25 (Kahn [1, 2])**.** *Let H be a d-regular linear k-graph. Then for all vertex $v \in V(H)$, we have $\mathcal{P}_H(\overline{v}) = (1 + o_d(1))d^{-1/k}$.*

This conjecture was verified for the case $k = 2$ by Kahn and Kim [3] in 1998.

Very recently, I disprove Conjecture 25 for all $k \geqslant 3$.

Theorem 26 (L., 2024+ [4])**.** *Let $k > 2$ be an integer and $d_0, \varepsilon > 0$ be a positive real number. Then there exist $d > d_0$ and d-regular linear k-graph H such that the following holds. There are two vertices $v_1, v_2 \in V(H)$ which satisfies the following.*

- $\mathcal{P}_H(\overline{v_1}) > 1 - \frac{1+\varepsilon}{d^{k-2}}$,
- $\mathcal{P}_H(\overline{v_2}) < \frac{1+\varepsilon}{d+1}$.

I also show that the gap between two probabilities in Theorem 26 is the best possible.

Theorem 27 (L., 2024+ [4])**.** *Let $k > 2$ be an integer and $\varepsilon > 0$ be a real number. Then there exists $d_0 > 0$ such that for all $d \geqslant d_0$ and for all d-regular linear k-graph H, the following inequality holds for all $v \in V(H)$.*

$$\frac{1}{d+1} \leqslant \mathcal{P}_H(\overline{v}) < 1 - \frac{1-\varepsilon}{d^{k-2}}.$$

The proof of Theorem 26 use limits of dynamical systems, so our hypergraph is very large compared with d. Then what if d also varies with the order of the hypergraph? In this case, we may expect Conjecture 25 would be true. Hence, I raised the following conjecture.

Conjecture 28 (L., 2024+ [4])**.** *Let $k \geqslant 3$ be a positive integer and $0 < \delta, \varepsilon < 1$ be a positive real number. Then there is n_0 such that the following is true for all $n \geqslant n_0$. Let H be an n-vertex k-graph that satisfies the following. For all $v \in V(H)$, the degree of v lies between $(1 - n^{-\delta})d$ and $(1 + n^{-\delta})d$ for some $d > n^{\varepsilon}$, and the maximum codegree of H is bounded above by $n^{-\delta}d$. Then for all $v \in V(H)$, we have*

$$\mathcal{P}_H(\overline{v}) = (1 + o_n(1))d^{-1/k}.$$

References

[1] Jeff Kahn. “Asymptotics of hypergraph matching, covering and coloring problems". In: *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*. Birkhäuser, Basel, 1995, pp. 1353–1362. ISBN: 3-7643-5153-5.

[2] Jeff Kahn. “On some hypergraph problems of Paul Erdős and the asymptotics of matchings, covers and colorings". In: *The mathematics of Paul Erdős, I. Vol. 13*. Algorithms Combin. Springer, Berlin, 1997, pp. 345–371. ISBN: 3-540-61032-4. DOI: 10.1007/978-3-642-60408-9_26.

[3] Jeff Kahn and Jeong Han Kim. “Random matchings in regular graphs". In: *Combinatorica* 18.2 (1998), pp. 201–226. issn: 0209-9683,1439-6912. doi: 10.1007/PL00009817.

[4] Hyunwoo Lee. “Random matchings in linear hypergraphs". *arxiv:2406.06421*

The α-edge-crossing width

Myounghwan Lee
Hanyang University

The α-edge-crossing width is introduced by Chang, Kwon, and Lee. Roughly, this parameter minimizes the maximum size of bags when the number of edges crossing a bag is bounded by α. The relationship between α-edge-crossing width and other graph parameters is as follows.

$$\begin{tikzcd}
 & & \mathrm{fen} \arrow[d] \\
\mathrm{carvw} \arrow[dr] \arrow[dd] & & \mathrm{ecw} \arrow[dl] \\
 & \mathrm{stcw} \arrow[dl] \arrow[dr] & \\
\mathrm{tcw} \arrow[dr] & & \mathrm{ecrw}_\alpha \arrow[dl] \\
 & \mathrm{ecrw} \sim \mathrm{tpw} \arrow[d] & \\
 & \mathrm{tw} &
\end{tikzcd}$$

FIGURE 0.1. The hierarchy of the mentioned width parameters. For two width parameters A and B, $A \to B$ means that every graph class of bounded A has bounded B, but there is a graph class of bounded B and unbounded A. Also, $A \sim B$ means that two parameters A and B are equivalent. fen, carvw, ecw, tcw, stcw, ecrw_α, ecrw, tpw, and tw denote feedback edge set number, carving-width, edge-cut width, tree-cut width, slim tree-cut width, α-edge-crossing width, edge-crossing width, tree-partition-width, and tree-width, respectively.

Also, it is known that

- twin-width is closed under mixed minor,
- tree-width is closed under graph minor, and
- tree-cut width is closed under weak immersion, etc.

Question 29. *For fixed α, does there exist some operation (except vertex and edge deletion) such that α-edge-crossing width is closed under that operation?*

If we find such an operation, then we have one more question.

Question 30. *Does there exist some graph class $\mathcal{H}$ such that a graph G has large α-edge-crossing width iff G has some $H \in \mathcal{H}$ as such operation with vertex and edge deletion?*

Partitioning the hypercube into smaller subcubes

Yongjin Lee

UIUC

Given a d-dimensional hypercube Q_d, let $f(d)$ be the number of ways to partition Q_d into subcubes. More generally, for $S \subseteq [d] \cup \{0\}$, we denote by $f_S(d)$ the number of ways to partition Q_d into subcubes such that each subcube has a dimension belongs to S. For instance, $f_{\{1\}}$ is the number of perfect matchings of Q_d, whereas $f_{\{0,1\}}$ equals to the number of different matchings of Q_d. I will introduce some results by Alon, Balogh and Potapov [1] on asymptotic of these functions for some interesting $S \subseteq [d] \cup \{0\}$, as well as some open problems.

References

[1] N. Alon, J. Balogh, and V. Potapov, *Partitioning the hypercube into smaller hypercubes*, arXiv:2401.00299.

Realizing a Toeplitz graph

Homoon Ryu

Seoul National University

An $n \times n$ matrix $A = (a_{ij})$ is called *Toeplitz*, if $a_{ij} = a_{i+1,j+1}$ for any $i, j \in \{1, 2, \ldots, n-1\}$. A *Toeplitz graph* is a graph whose adjacency matrix is Toeplitz. If the vertices are labeled, then it is easy to check whether its adjacency matrix is Toeplitz or not. However, if the vertices is not labeled, it is not easy to determine that it is Toeplitz or not. In this context, I propose the following problem.

Problem 31. *Is there a proper algorithm to determine that a given graph is Toeplitz or not? This question can be rewrite as follows: For a given square matrix A, is there a permutation matrix P such that makes $P^T AP$ Toeplitz?*

Monochromatic connectivity of graphs

Jaehyeon Seo

Yonsei University

An edge-colored path is said to be *monochromatic* if all of its edges have the same color. For a connected graph G, the *monochromatic connection number* of G, denoted by $mc(G)$, is the maximum number of colors in an edge-coloring of G so that any two vertices are connected by a monochromatic path. This was introduced in [3]. See [4] for a survey related to this concept. By generalizing this for graphs with higher vertex-connectivity, the *monochromatic k-connection number* $mc_k(G)$ was introduced in [5], which is defined as the maximum number of colors in an edge-coloring of G so that any two vertices are connected by k internally vertex-disjoint monochromatic paths. Let

$$h_k(G) = \max\{mc_k(H) : H \text{ is a minimum spanning } k\text{-connected subgraph of } G\}.$$

Let H be a minimum spanning k-connected subgraph of G. Color $E(H)$ so that H has a monochromatic k-connected coloring with $h_k(G)$ colors, and color all the edges of $E(G) \smallsetminus E(H)$ by further distinct colors. This gives a monochromatic k-connected coloring of G, whence we may observe $mc_k(G) \geqslant e(G) - e(H) + h_k(G)$. The following conjecture claims that such a coloring of G attains $mc_k(G)$.

Conjecture 32 ([5, Conjecture 3])**.** *Let G be a k-connected graph, where $k \geqslant 2$, and let $H \subseteq G$ be a minimum spanning k-connected subgraph. Then*

$$mc_k(G) = e(G) - e(H) + h_k(G)$$

In particular, if $G = K_n$, Conjecture 32 becomes the following.

Conjecture 33 ([5, Conjecture 21])**.** *Let $n > k \geqslant 2$. Then*

$$mc_k(K_n) = \binom{n}{2} - \left\lceil \frac{kn}{2} \right\rceil + 1.$$

The following is a partial result of Conjecture 33.

Theorem 34 ([5, Theorem 22])**.** *We have the following.*

(1) Let $n > k \geqslant 2$. Then

$$\binom{n}{2} - \left\lceil \frac{kn}{2} \right\rceil + 1 \leqslant mc_k(K_n) \leqslant \binom{n}{2} - \left\lceil \frac{(k-1)n(n-1)}{2(n-2)} \right\rceil + 1.$$

(2) Let $n > k$, where $k = 2, 3, 4, 5$. Then

$$mc_k(K_n) = \binom{n}{2} - \left\lceil \frac{kn}{2} \right\rceil + 1.$$

The parameters $mc(G)$ and $mc_k(G)$ are natural opposites of the *rainbow connection number $rc(G)$* and the *rainbow k-connection number $rc_k(G)$*, respectively, which were introduced in [1] and [2], respectively. I may propose a general question, which seeks for good connections between the monochromatic parameters and the rainbow parameters.

References

[1] G. Chartrand, G. L. Johns, K. A. McKeon, and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. 133 (2008) 85–98.

[2] G. Chartrand, G.L. Johns, K. A. McKeon, and P. Zhang, *The rainbow connectivity of a graph*, Networks 54 (2009) 75–81.

[3] Y. Caro and R. Yuster, *Colorful monochromatic connectivity*, Discrete Math. 311 (2011) 1786–1792.

[4] X. Li and D. Wu, *A Survey on Monochromatic Connections of Graphs*, Theory and Applications of Graphs: Vol. 0: Iss. 1, Article 4. (2018)

[5] Q. Cai, S. Fujita, H. Liu, and B. Park, *Monochromatic k-connection of graphs*, arXiv:2402.09254.

Graph class for Jones' conjecture

Hyeonjun Shin

POSTECH

For a graph $G = (V, E)$, a Cycle Packing of G is a set of vertex-disjoint cycles of G, and a Feedback Vertex Set is a set of vertices of G such that the graph induced by the vertices not in a feedback vertex set is a forest. For the optimization version of these problems, we consider a maximum cycle packing and a minimum feedback vertex set. For a graph G, let $\mathsf{CP}(G)$ denote the cardinality of a maximum cycle packing, and let $\mathsf{FVS}(G)$ denote the cardinality of a minimum feedback vertex set. There is a well-known conjecture of the relation between the optimal size of the cycle packing and feedback vertex set of a planar graph, called *Jones' conjecture*.

Conjecture 35. [1, Jones' conjecture] *For every planar graph G. $\mathsf{FVS}(G) \leqslant 2\mathsf{CP}(G)$*

$\mathsf{FVS}(G) \leqslant 5\mathsf{CP}(G)$ is proven by Kloks et al. [1, Theorem 8] in 2002, and Chen et al [2], Ma et al. [3], and Chappell et al [4] proved $\mathsf{FVS}(G) \leqslant 3\mathsf{CP}(G)$, independently. Furthermore, for some graph classes, Jones' conjecture is proven.

- Wheel graphs by Kloks et al. [1] in 2002,
- The special case of outerplanar graphs by Kloks et al. [1, Theorem 11] in 2002,
- Subcubic graphs by Bonamy et al [5] in 2019,
- Halin graphs (based planar graphs) by Bärnkopf and Győri [6] in 2024.

A *based planar graph* is a planar graph that has a face that is adjacent to every other face. A *Halin graph* is a planar graph constructed by connecting the leaves of a tree into a cycle, and a Halin graph is a based planar graph. In [6], they prove Jones' conjecture for a based planar graph as follows. First, they show that in such a based planar graph, there is a triangle xyz such that $d(x) = d(y) = 3$ and x and y are on the outer face. And, they prove Jones' conjecture by induction, and they prove the inductive step in their proof as follows. First, if there is a triangle xyz such that $d(x) = 2$, then they prove the induction by deleting y and z. If there is a vertex of degree two not in a triangle, then they contract the vertex (traditional reduction rule of kernelization for FVS problem). The resulting graph is a based planar graph of which every vertex has a degree of at least three, and thus there is a triangle xyz such that $d(x) = d(y) = 3$ and x and y are on the outer face. Then, they prove the induction by deleting y and z.

Recall that in a based planar graph, a face is adjacent to every other face. Thus, naturally, the authors propose proving Jones' conjecture for a planar graph in which there is a cycle which has at least one edge in common with every face. And, similarly, they propose proving Jones' conjecture for a Hamiltonian planar graph.

Problem 36. [6, Problem 3.1] *Let be G a planar graph in which there is a cycle which has at least one edge in common with every face. Then, $\mathsf{FVS}(G) \leqslant 2\mathsf{CP}(G)$?*

Problem 37. [6, Problem 3.2] *Let be G a Hamiltonian planar graph. Then, $\mathsf{FVS}(G) \leqslant 2\mathsf{CP}(G)$?*

References

[1] T. Kloks, C-M Lee, and J Liu. "New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs", *International Workshop on Graph-Theoretic Concepts in Computer Science*, 282-295, 2002.

[2] H-B Chen, H-L Fu, and C-H Shih. "Feedback vertex set on planar graphs", *Taiwanese Journal of Mathematics*, 16(6):2077-2082, 2012.

[3] J. MA, X. YU, and W. Zang. "Approximate min-max relations on plane graphs", *Journal of Combinatorial Optimization*, 26(1):127-134, 2013.

[4] G. G. Chappell, J. Gimbel, and C. Hartman. "On cycle packings and feedback vertex sets", *Contributions to Discrete Mathematics*, 9(2):17-34, 2014.

[5] M. Bonamy, F. Dross, T. Masařík, W. Nadara, M. Pilipczuk, and M. Pilipczuk. "Jones' Conjecture in subcubic graphs", `arXiv:1912.01570`.

[6] P. Bärnkopf, and E Győri. "Jones' conjecture for Halin graphs and a bit more", `arXiv:2401.07376`.

Estimating the size of k-Diophantine tuple over $\mathbb{F}_q$

Semin Yoo

IBS DIMAG

In this project, we are interested in finding the largest size of k-Diophantine tuple over $\mathbb{F}_q$.

Definition 38. Let S be a set of m positive integers $\{a_1, a_2, \ldots, a_m\}$. If $a_{i_1} a_{i_2} \cdots a_{i_k} + 1$ is a perfect square for all $i_1, \ldots, i_k \in \{1, 2, \ldots, m\}$ such that $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$, then S is called a k-**Diophantine tuple**.

Let $M_k(\mathbb{F}_q)$ be the maximum size of k-Diophantine tuples over $\mathbb{F}_q$. In 2023, Hammonds, Kim, Miller, Nigam, Onghai, Saikia, and Sharma [1, Theorem 1.3] confirmed $M_k(\mathbb{F}_q) \geqslant \Theta((\log q)^{1/(k-1)})$. In fact, they only considered the case where q is an odd prime, but the same proof extends to all odd prime powers q. Later, Yip and Y. [2] significantly improved their bound as follows.

Theorem 39. *Let $k \geqslant 2$ and let q be an odd prime power. There is an k-Diophantine tuple over $\mathbb{F}_q$ with size at least $\left(\frac{1}{k} - o(1)\right) \log_4 q$, as $q \to \infty$.*

Question 40. *Can we further improve the lower bound above?*

One possible idea is to use the theory of hypergraphs. We can define a k-uniform hypergraph by using the property of k-Diophantine tuples. Can we borrow any techniques from the theory of hypergraphs to improve the size of k-Diophantine tuple over $\mathbb{F}_q$?

References

[1] T. Hammonds, S. Kim, S. J. Miller, A. Nigam, K. Onghai, D. Saikia, and L. M. Sharma. "k-Diophantine m-tuples in finite fields" *International Journal of Number Theory*, 19(4):891-912, 2023.

[2] C. H. Yip, and S. Yoo. "F-Diophantine sets over finite fields", `arXiv:2406.00310`.

LIST OF PARTICIPANTS

Name	Affiliation	E-mail address
김동규	KAIST & IBS DIMAG	donggyu@kaist.ac.kr
김석범	KAIST & IBS DIMAG	seokbeom@kaist.ac.kr
김주원	KAIST	juwon35721@kaist.ac.kr
류호문	Seoul National University	rhm95@snu.ac.kr
서재현	Yonsei University	jaehyeonseo@yonsei.ac.kr
신현준	POSTECH	tlsguswns119@postech.ac.kr
안신우	POSTECH	shinwooan@postech.ac.kr
안정호	KIAS	junghoahn@kias.re.kr
유세민	IBS DIMAG	syoo19@ibs.re.kr
이도현	KAIST & IBS DIMAG	mathloveguy@kaist.ac.kr
이명환	Hanyang University	kolas22@naver.com
이용진	UIUC	yongjin4@illinois.edu
이현우	KAIST & IBS ECOPRO	hyunwoo.lee@kaist.ac.kr
임성혁	KAIST & IBS ECOPRO	seonghyuk@kaist.ac.kr
장연수	Hanyang University	yeonsu@hanyang.ac.kr
조경진	POSTECH	kyungjincho@postech.ac.kr
최무진	KAIST & IBS DIMAG	mujinchoi@kaist.ac.kr
추호진	Seoul National University	ghwls8775@snu.ac.kr